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Singularities in Spacetimes with Torsion

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The geometry of timelike congruences in spacetimes with torsion is considered. An extension of Hawking's cosmological singularity theorem is proposed and a comparison with the general relativity results is given.

1. INTRODUCTION

The investigation of the influence of singularities on the geometry of spacetime led Hawking (1966) to formulate several theorems on the convergence of timelike geodesic congruences, which later (Hawking and Penrose, 1970) proved to be useful in the study of the occurrence of singularities in cosmology (Hawking, 1966). Some years later two different groups led by F. W. Hehl and A. Trautman laid the foundations of cosmology based on the Einstein–Cartan–Sciama–Kibble (ECSK) theory. The main hope of the Trautman group was to establish a singularity-free cosmology where the spin–spin contact interaction would be the agent responsible for avoiding gravitational collapse. The Trautman theory considered a spinning fluid in a spacetime with torsion, in contrast to what happened in general relativity, where the gravitational collapse could not be avoided regardless of symmetry constraints on the cosmological models (Penrose, 1965). Although some cosmological models in ECSK have been constructed (Trautman, 1973; Tafel, 1973; Kopczynski, 1972), some singular models have also been constructed. In particular, Kerlick (1975) has shown, by studying the Dirac field as a source for the torsion, that it causes a positive effective mass term in the energy condition of the generalized singularity theorem for ECSK, making use of Hehl's energy-momentum tensor, where the

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spin–spin contact interaction enhances rather than opposes singularity formation (see also Bedran and Garcia de Andrade, 1983). Based on these facts, it seems that the singularity problem in ECSK is still an open question in gravitational physics and deserves further investigation. Here I propose the germ of such a program by considering the behavior of the timelike nongeodesic autoparallel congruences in ECSK. An expression for the Raychauduri equation in Riemann–Cartan spacetime is also derived and the consequences for the singularity theorem are discussed.

2. THE GEOMETRY OF TIMELIKE CONGRUENCES IN SPACETIMES WITH TORSION

The physical meaning of the word singularity is sometimes associated with divergences as in the case of Maxwell fields. Nevertheless, Geroch (1967, 1968*a,b*) has shown that the situation is not so simple, and a detailed study of singularities would require some knowledge of the topology of general relativity (Penrose, 1972) and the study of the completeness of geodesics. One of the main points in the analysis of Hehl's group of the singularity problem extension to ECSK was the assumption that the causality structure of spacetimes with torsion would be the same as the Riemannian one, i.e., that the torsion does not couple with null geodesics (like those of photons, neutrinos, or gravitons). Nevertheless, some authors (Prasanna, 1975; de Sabbata and Gasperini, 1981) consider that photons do couple with torsion and that therefore the causality structure in U_4 should be distinct from that of the Riemannian spacetime of general relativity. Here we show that the convergence of timelike curves in Riemann–Cartan spacetime is still valid in case this timelike vector is the four-velocity of the spinning fluid. However, the situation is not so simple in the case ξ^a $(a = 1, 2, 3, 4)$ is an arbitrary timelike vector field describing a general timelike congruence in U_4. Let us now consider (Geroch, 1970) the convergence $c = -\nabla_a \xi^a$, where ∇_a is the Riemann–Cartan connection. Then

$$\xi^b \nabla_b c = -\xi^b \nabla_b \nabla_a \xi^a = -\nabla_a(\xi^b \nabla_b \xi^a) + \nabla_a \xi_b \nabla^b \xi^a + R_{ab} \xi^a \xi^b - 2S^b_{ad} \xi^d \nabla_b \xi^a \tag{2.1}$$

where we have used the Ricci identity in the Riemann–Cartan spacetime (Schouten, 1954)

$$2\nabla_{[d} \nabla_{c]} \xi_a = R^b_{adc} \xi_b - 2S^b_{cd} \nabla_b \xi_a$$

where R^b_{adc} is the Riemann tensor in U_4 and S^b_{cd} is the torsion tensor. In the case of a spinning fluid congruence, $\xi^a = u^a$, where u^a $(u^a u_a = -1)$ is the four-velocity of the fluid, the Frenkel condition $S^c_{ab} = S_{ab} u^c$, $S^b_{ab} = 0$, reduces expression (2.1) to

$$u^b \nabla_b c = \nabla_a u_b \nabla^b u^a + R_{ab} u^a u^b \tag{2.2}$$

This equation reduces to the Raychauduri equation for the spinning fluid (Fennelly *et al.*, 1986)

$$R_{ab} u^a u^b = -\tfrac{1}{3}\theta^2 + \sigma^2 + \omega^2 + S^2 \tag{2.3}$$

where we have used the decomposition of the covariant derivative of the four-velocity in terms of the fluid parameters, shear σ^2, angular velocity ω^2, and expansion θ^2,

$$\nabla_a u_b = \omega_{ab} + \sigma_{ab} + \tfrac{1}{3}\theta h_{ab} + A_a u_b \tag{2.4}$$

where ω_{ab}, σ_{ab}, h_{ab}, and A_b are, respectively, the angular velocity, shear, and projection tensors and the acceleration of the spinning fluid. Let us now consider the case of a shear-free, irrotational cosmological model. In this case the Raychauduri equation reduces to

$$R_{ab} u^a u^b = S^2 - \tfrac{1}{3}\theta^2 \tag{2.5}$$

From the difference in sign in front of the terms S^2 and θ^2 one can see that the stage of evolution of the universe where the spin density $S^2 = S_{ab}S^{ab}$ dominates the expansion ($S^2 \geq \frac{1}{3}\theta^2$), which means physically in the state where the spin density $S^2 = S_{ab}S^{ab}$ dominates the expansion, then

$$R_{ab}(\Gamma) u^a u^b \geq 0 \tag{2.6}$$

which is exactly the analog of the Hawking convergence condition in U_4, since here the Ricci tensor is computed in terms of the Cartan connection. In fact, it reduces to the Hawking formula $R_{ab}(\{\cdot\})u^a u^b \geq 0$ for the convergence of geodesics in torsion-free spacetimes. Here $\{\cdot\}$ represents the symmetric Levi-Civita-Christoffel symbol. In the case of static spacetimes with torsion (Bedran and Garcia de Andrade, 1983) $\theta^2 = \omega^2 = \sigma^2 = 0$ and the convergence formula has a lower bound $R_{ab}(\Gamma)u^a u^b = S^2 \geq 0$. All these facts lead us to conclude that we should consider expression (2.6) as a candidate for the extension of the Hawking-Penrose singularity theorem to spacetimes with torsion. We follow the Geroch argument path in U_4. First we shall consider timelike congruences that obey the autoparallel (Hehl *et al.*, 1976) equation

$$\xi^b \nabla_b \xi^a = 0 \tag{2.7}$$

In the case of torsion-free connections, (2.7) reduces to the geodesic equation $\xi^b \nabla_b^{\{\}} \xi^a = 0$. With (2.7) we can say that $\nabla_a \xi_b$ is a symmetric tensor in the 3-space orthogonal to ξ^a. This also simplifies equation (2.1) to

$$\xi^b \nabla_b c = (\nabla_a \xi_b)(\nabla^b \xi^a) + R_{ab}(\Gamma)\xi^a \xi^b - 2S^b_{\ ad}\xi^d \nabla_b \xi^a \tag{2.8}$$

From the matrix inequality trace $A^2 \geq \frac{1}{3}$ (trace $A)^2$ for a 3×3 matrix A we obtain

$$(\nabla_a \xi_b)(\nabla^b \xi^a) \geq \tfrac{1}{3} c^2 \tag{2.9}$$

From (2.9) we cannot say much about the behavior of the timelike congruence, but if the torsion tensor S^b_{ad} is a constraint to the expression

$$R_{ab}(\Gamma)\xi^a \xi^b \geq 2 S^b_{ad} \xi^d \nabla_b \xi^a \tag{2.10}$$

then (2.8)-(2.10) imply

$$\xi^b \nabla_b c \geq \tfrac{1}{3} c^2 \tag{2.11}$$

Physically this means that the congruence of autoparallels in U_4 converges. This condition reduces to Hawking's convergence condition in GR. However in GR, condition (2.11) is equivalent to the strong energy condition

$$(T^{\mathrm{GR}}_{ab} - \tfrac{1}{2} g_{ab} T^{\mathrm{GR}}) \xi^a \xi^b \geq 0 \tag{2.12}$$

via the Einstein field equations, where T_{ab} is the stress-energy tensor. Here a similar situation occurs if one considers the quasi-Einsteinian form of the Einstein-Cartan field equations

$$R_{ab}(\Gamma) = (T^{\mathrm{ECSK}}_{ab} - \tfrac{1}{2} g_{ab} T^{\mathrm{ECSK}}) \tag{2.13}$$

Assuming that our vector ξ^a coincides with the four-velocity of the spinning fluid, and using the Frenkel condition $S_{ad} u^d = 0$ in (2.11), yields

$$R_{ab}(\Gamma) u^a u^b \geq 0 \tag{2.14}$$

Expression (2.14) can be considered as an extension of Hawking's convergence condition to spacetimes with torsion and is therefore a good candidate for the extension of Hawking-Penrose theorems to Riemann-Cartan spacetime. It is also important to mention that in this case the condition (2.14) is equivalent to the strong energy condition in spacetimes with torsion.

3. DISCUSSION

A natural consequence of the study made here could be to investigate the energy conditions, weak, strong, and dominant, in ECSK and their connection with the geometry of the singularities. Also, the causal behavior of autoparallels should be investigated along the same lines as Penrose did in GR (Penrose, 1972). This appears elsewhere (Garcia de Andrade, 1990).

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